ASYMPTOTIC APPROXIMATION OF THE MODAL ACOUSTIC IMPEDANCE OF A CIRCULAR MEMBRANE

WOJCIECH P. RDZANEK*, WITOLD J. RDZANEK and KRZYSZTOF SZEMELA
Department of Acoustics, Institute of Physics
University of Rzeszów, Al. Rejtana 16A
35-310 Rzeszów, Podkarpackie, Poland
*wprdzank@uniw.rzeszow.pl

Received 20 July 2009
Revised 9 April 2010

The Neumann boundary value problem of the Helmholtz equation of a vibrating circular membrane embedded into a flat rigid baffle is solved. The membrane is excited asymmetrically and radiates acoustic waves into the half-space above the baffle. A set of elementary asymptotic equations for modal radiation self-impedance and mutual impedance is presented. The equations are necessary for numerical computations of the radiated active and reactive acoustic power including the acoustic attenuation. A few equations available in the literature are collected. All the missing equations have been obtained using the methods of analysis of contour integral and stationary phase. The presented equations cover a wide frequency band, with the exception of the lowest frequencies and the frequencies close to coincidence.

Keywords: Modal acoustic impedance; acoustic attenuation; Green function; asymmetric asymptotics; time harmonic steady-state vibrations; eigenfunctions; eigenfrequencies; sound radiation.

1. Introduction

A number of casings for industrial as well as common use devices consist of flat surfaces which vibrate and radiate acoustic waves. Some elementary equations are useful at the initial stage of designing this type of casings because they allow fast numerical calculations related to casings’ vibroacoustic properties such as acoustic power and acoustic impedance. Vibrating rectangular and circular pistons are the simplest models of flat radiators. They were used to obtain formulas for calculation of the radiation impedance. The obtained results vary in precision from exact to approximate. The advantage of using a vibrating piston as a model is the simplicity of its analysis. The disadvantage is its far-from-perfection compliance with the actual behavior of vibrating surface elements, often being subjected to elastic deformation due to vibration. Therefore, vibrating membranes and thin plates are better models. However, the calculation of radiation impedance becomes more complex than in the case of piston radiators. In general, the exact formulas cannot be obtained and therefore, various approximate methods are widely used with respect to both vibrations...
and sound radiation. The Rayleigh-Ritz method as well as a number of other estimation methods are often used\textsuperscript{5–14} to calculate approximate values of the eigenfrequencies and other vibroacoustic quantities. Several discrete calculation methods have often been hybridized with the vibration velocity measurements to determine the sound radiation efficiency of vibrating structures within the low frequency range.\textsuperscript{15–17} While the finite element method, the boundary element method, the lumped parameter model and the radiation resistance matrix have often been used to evaluate numerically both the vibration velocity and the sound radiation properties of different flat deformable bodies\textsuperscript{18–21}; a number of approximate methods such as the discontinuous Galerkin method, the nonconforming finite element, discrete Fourier transform modal analysis of spectral element method have been used for solving numerically the wave equation and determining the acoustic waves propagated.\textsuperscript{22–25}

In a few cases, the calculation of radiation impedance has been conducted on the basis of exact solutions to the vibration problems of thin flat circular, rectangular and annular membranes and plates.\textsuperscript{26–31} However, the results mentioned relate to the radiation of acoustic waves mainly for axisymmetric modes of vibrating deformable circular and annular radiators embedded into a flat rigid baffle. Taking into account both axisymmetric and asymmetric modes leads to a significant number of modal impedance values to be included in the calculation of vibroacoustic quantities such as vibration velocity, acoustic pressure and acoustic power including acoustic attenuation. The use of integral equations leads to a significantly time-consuming numerical calculations and difficulties in the vicinity of singular points appearing on the integration contour. Therefore, asymptotic formulas are particularly useful. So far, the asymptotic formulas of modal self- and mutual- radiation impedance have been presented for circular and annular membranes as well as for thin plates. These formulas are valid for the frequencies above the coincidence for the axisymmetric problem\textsuperscript{26, 32, 33} as well as for the modal radiation self-reactance of the clamped circular plate below the coincidence and also for the axisymmetric problem.\textsuperscript{34} However, the formulas presented in the latter study carry a considerable approximation error. Up to now, asymptotic formulas have not been derived for the calculation neither of modal radiation self resistance nor of the mutual impedance below the coincidence frequency, nor for frequencies between the two intermediate coincidence frequencies in the axisymmetric case. Similarly, the asymptotic formulas of the impedance of a vibrating circular membrane in the case of sound radiation related to the mode pair where at least one of the modes is asymmetric have not yet been presented either. This paper presents all the missing asymptotic formulas of the modal radiation impedance of a vibrating circular membrane that are useful for fast numerical analyses of vibroacoustic steady state processes together with the asymmetric excitation and acoustic attenuation. Also it has been shown how to use the presented modal quantities to obtain some proper result for the corresponding physical quantities such as acoustic power, acoustic pressure and vibration velocity including acoustic attenuation.
2. Physical Quantities, Completeness of Modal Quantities and the Coupling Matrix

Often the same two questions arise: What is the modal acoustic impedance and how can it be used for calculations of measurable physical quantities such as acoustic power, acoustic pressure and vibration velocity including the acoustic attenuation? Therefore, this section is devoted to providing answers to the following two questions. How to ensure that the calculations are conducted properly? How to ensure the completeness\(^3\) of the corresponding modal quantities?

The Green function being the solution to the Neumann boundary value problem of the Helmholtz equation\(^{36,37}\) has been used for \(z_0 = 0\)

\[
G(\mathbf{r} | \mathbf{r}_0) = \frac{i}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_0)\} \frac{d\xi d\eta}{\gamma} \tag{1}
\]

where \(\gamma^2 = k^2 - \xi^2 - \eta^2\), \(\mathbf{k} = (\xi, \eta, \gamma)\) is the wavevector in the Cartesian coordinates and its length \(k = |\mathbf{k}|\) is the acoustic wavenumber, \(\mathbf{r} = (x, y, z)\) and \(\mathbf{r}_0 = (x_0, y_0, 0)\) are the leading vectors of the field point and the acoustic wave source point, correspondingly, whereas \(r = |\mathbf{r}|\) and \(r_0 = |\mathbf{r}_0|\) are their lengths. It has been assumed that the acoustic pressure is time-harmonic \(p(\mathbf{r}, t) = p(\mathbf{r})e^{-i\omega t}\) and its amplitude has been formulated as

\[
p(\mathbf{r}) = -ik\varrho c \int_{S_0} v_N(\mathbf{r}_0) G(\mathbf{r} | \mathbf{r}_0) dS_0 = \frac{1}{4\pi^2} k\varrho c S_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(i\mathbf{k} \cdot \mathbf{r}) M(\mathbf{k}) \frac{d\xi d\eta}{\gamma} \tag{2}
\]

where \(v_N(\mathbf{r}_0)\) is the normal component of the membrane’s vibration velocity vector, \(S_0 = \pi a^2\) is the membrane’s area,

\[
M(\mathbf{k}) = \frac{1}{S_0} \int_{S_0} v_N(\mathbf{r}_0) \exp(-i\mathbf{k} \cdot \mathbf{r}_0) dS_0 \tag{3}
\]

is the membrane’s radiation function, i.e. \(D = |M(\mathbf{k})|/\text{Max}_{\phi,\varphi} |M(\mathbf{k})|\) is its directivity pattern and \(\mathbf{k} = (k, \phi, \varphi)\) is the wavevector in the spherical coordinates. It has been assumed that the membrane’s vibrations are time-harmonic and the normal component of vibration velocity is \(v_N(\mathbf{r}_0, t) = v_N(\mathbf{r}_0)e^{-i\omega t} = -i\omega W(\mathbf{r}_0, t)\), where \(W(\mathbf{r}_0, t) = W(\mathbf{r}_0)e^{-i\omega t}\) is the transverse deflection. The amplitude has been formulated as the complete eigenfunction system series

\[
W(\mathbf{r}_0) = W(r_0, \varphi_0) = \sum_m \sum_n \left\{ c_{m,n}^{(c)} W_{m,n}^{(c)}(r_0, \varphi_0) + c_{m,n}^{(s)} W_{m,n}^{(s)}(r_0, \varphi_0) \right\} \tag{4}
\]

where the summation applies within the limits \(m = 0, \ldots, \infty\) and \(n = 1, \ldots, \infty\). The eigenfunctions of the corresponding modes \((m, n)\) degenerate to the cosine and sine mode.
pairs for $m \geq 1$
\[
\begin{align*}
\begin{pmatrix}
W_{m,n}^{(c)}(r_0, \varphi_0) \\
W_{m,n}^{(s)}(r_0, \varphi_0)
\end{pmatrix} &= W_{m,n}(r_0) \begin{pmatrix}
\cos m \varphi_0 \\
\sin m \varphi_0
\end{pmatrix}, \\
W_{m,n}(r_0) &= \sqrt{\varepsilon_m} \frac{J_m(k_{m,n} r_0)}{J_{m+1}(\beta_{m,n})}
\end{align*}
\]
where $m$ and $n$ are the numbers of nodal diameters and nodal circles, respectively, $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \geq 1$, $\beta_{m,n} = k_{m,n} a$ is the root of frequency equation of the membrane $J_m(\beta_{m,n}) = 0$, $k_{m,n} = \omega_{m,n}/c_M$ is the structural wavenumber of the mode $(m, n)$, $\omega_{m,n}$ is the eigenfrequency of the mode, $c_M = \sqrt{T/\sigma}$ is the wave propagation velocity, $T$ is the membrane’s tension and $\sigma$ is the membrane’s mass per surface unit. The eigenfunctions satisfy the orthogonality condition
\[
\frac{1}{S_0} \int_{S_0} \begin{pmatrix}
W_{m,n}^{(c)}(r_0) \\
W_{m,n}^{(s)}(r_0)
\end{pmatrix} \begin{pmatrix}
W_{m',n'}^{(c)}(r_0) \\
W_{m',n'}^{(s)}(r_0)
\end{pmatrix} dS_0 = \delta_{m,m'} \delta_{n,n'} \begin{pmatrix}
1 \\
\text{sign}_{m,m'}
\end{pmatrix}
\]
where $\delta_{m,m'}$ is the Kronecker delta, $\text{sign}_{m,m'} = \text{sign}_m \text{sign}_{m'}$ and $\text{sign}_m$ is the signum function with integer argument $m$. The eigenfunctions from Eq. (5) are the solutions to the following equation of motion of the membrane’s free vibrations in vacuum
\[
(k_{m,n}^2 \nabla^2 + 1) \begin{pmatrix}
W_{m,n}^{(c)}(r_0) \\
W_{m,n}^{(s)}(r_0)
\end{pmatrix} = 0
\]
where $\nabla^2 = (1/r_0)(\partial/\partial r_0)[r_0(\partial/\partial r_0)] + (1/r_0^2)(\partial^2/\partial \varphi_0^2)$ is the Laplacian in polar coordinates. In this case the membrane vibrates with all its eigenfrequencies $\omega_{m,n}$. The polar coordinates have been introduced for both the field point and the source point
\[
x = r \cos \varphi, \quad y = r \sin \varphi; \quad x_0 = r_0 \cos \varphi_0, \quad y_0 = r_0 \sin \varphi_0
\]
Equation (4) has been used to formulate the acoustic pressure and the radiation function as
\[
\begin{align*}
p(r) &= \sum_m \sum_n \frac{\omega}{\omega_{m,n}} \left\{ c_{m,n}^{(c)} p_{m,n}^{(c)}(r) + \text{sign}_m c_{m,n}^{(s)} p_{m,n}^{(s)}(r) \right\} \\
M(k) &= \sum_m \sum_n \frac{\omega}{\omega_{m,n}} \left\{ c_{m,n}^{(c)} M_{m,n}^{(c)}(k) + \text{sign}_m c_{m,n}^{(s)} M_{m,n}^{(s)}(k) \right\}
\end{align*}
\]
in terms of corresponding modal quantities given below
\[
\begin{align*}
\begin{pmatrix}
p_{m,n}^{(c)}(r) \\
p_{m,n}^{(s)}(r)
\end{pmatrix} &= k \frac{1}{4\pi^2} \varepsilon_0 c \int_{-\infty}^{+\infty} \int \exp(i \mathbf{k} \cdot \mathbf{r}) \begin{pmatrix}
M_{m,n}^{(c)}(k) \\
M_{m,n}^{(s)}(k)
\end{pmatrix} \frac{d\xi d\eta}{\gamma}
\end{align*}
\]
\[
\begin{align*}
\begin{pmatrix}
M_{m,n}^{(c)}(k) \\
M_{m,n}^{(s)}(k)
\end{pmatrix} &= \frac{1}{S_0} \int_{S_0} \begin{pmatrix}
\psi_{m,n}^{(c)}(r_0) \\
\psi_{m,n}^{(s)}(r_0)
\end{pmatrix} \exp(-i \mathbf{k} \cdot \mathbf{r}_0) dS_0
\end{align*}
\]
\[
\begin{align*}
\begin{pmatrix}
\psi_{m,n}^{(c)}(r_0) \\
\psi_{m,n}^{(s)}(r_0)
\end{pmatrix} &= \psi_{m,n}(r_0) \begin{pmatrix}
\cos m \varphi_0 \\
\sin m \varphi_0
\end{pmatrix}, \quad \psi_{m,n}(r_0) = -i \omega_{m,n} W_{m,n}(r_0)
\end{align*}
\]
Further, the function from Eq. (10b) has been formulated as

\[
\begin{pmatrix}
M^{(c)}_{m,n}(k) \\
M^{(s)}_{m,n}(k)
\end{pmatrix} = \frac{2}{a^2} \int_0^a v_{m,n}(r_0) \begin{pmatrix}
F^{(c)}_m(r_0) \\
F^{(s)}_m(r_0)
\end{pmatrix} r_0 dr_0
\]

where \(v_{m,n}(r_0) = -i \omega_{m,n} W_{m,n}(r_0)\),

\[
\begin{pmatrix}
F^{(c)}_m(r_0) \\
F^{(s)}_m(r_0)
\end{pmatrix} = \frac{1}{2\pi} \int_0^{2\pi} \exp\{-i(\xi r_0 \cos \varphi_0 + \eta r_0 \sin \varphi_0)\} \begin{pmatrix}
\cos m\varphi_0 \\
\sin m\varphi_0
\end{pmatrix} d\varphi_0
\]

\[
= (-i)^m J_m(\tau r_0) \begin{pmatrix}
\cos m\alpha \\
\sin m\alpha
\end{pmatrix}
\]

and the following transformations of the wavevector to its polar coordinates have been introduced

\[
\xi = \tau \cos \alpha, \quad \eta = \tau \sin \alpha
\]

which has led to \(\tau^2 = \xi^2 + \eta^2\) and \(\gamma^2 = k^2 - \tau^2\) giving \(k = (k, \alpha, \tau)\). After inserting Eq. (12) into Eq. (11) the following elementary formulas have been obtained

\[
\begin{pmatrix}
M^{(c)}_{m,n}(k) \\
M^{(s)}_{m,n}(k)
\end{pmatrix} = (-i)^{m+1} \omega_{m,n} \sqrt{\varepsilon_m} \psi_{m,n}(\tau) \begin{pmatrix}
\cos m\alpha \\
\sin m\alpha
\end{pmatrix}
\]

where

\[
\psi_{m,n}(\tau) = \frac{2}{a^2} \int_0^a \frac{J_m(k m n r_0)}{J_{m+1}(\beta_{m,n})} J_m(\tau r_0) r_0 dr_0 = \frac{2\beta_m J_m(\tau a)}{\beta_{m,n}^2 - (\tau a)^2}
\]

The acoustic power radiated by vibrating membrane into the upper half-space \(z \geq 0\)

\[
\Pi = \frac{1}{2} \int_S p(r) v_N^*(r) dS
\]

has been calculated using the impedance approach where \(S = \pi a^2\) is the surface directly adjacent to the membrane, \(r\) is the leading vector of the field point located on the surface \(S\) and \(v_N^*(r) = i \omega \tilde{W}(r)\) is the value conjugate to vibration velocity of the acoustic particle directly adjacent to the membrane’s surface and located at the point indicated by \(r\). The acoustic pressure from Eq. (9a) and the transverse deflection from Eq. (4) have been used together with Eq. (16) yielding

\[
\Pi = \sum_{m,m',n,n'} \frac{\omega^2}{\omega_m \omega_{m',n'}} \left\{ c^{(c)}_{m,n} c^{(c)*}_{m',n'} \Pi^{(c,c)}_{m,n;m',n'} + \text{sign}_{m'c^{(c)}_{m,n}} c^{(c)*}_{m',n'} \Pi^{(c,c)}_{m,n;m',n'} + \text{sign}_{m'c^{(c)}_{m,n}} c^{(c)*}_{m',n'} \Pi^{(c,c)}_{m,n;m',n'} \right\}
\]

\[
+ \text{sign}_{m'c^{(c)}_{m,n}} c^{(c)*}_{m',n'} \Pi^{(s,c)}_{m,n;m',n'} + \text{sign}_{m'c^{(c)}_{m,n}} c^{(c)*}_{m',n'} \Pi^{(s,c)}_{m,n;m',n'} + \text{sign}_{m'c^{(c)}_{m,n}} c^{(c)*}_{m',n'} \Pi^{(s,c)}_{m,n;m',n'} \right\}
\]

\[
+ \text{sign}_{m'c^{(c)}_{m,n}} c^{(c)*}_{m',n'} \Pi^{(s,s)}_{m,n;m',n'} + \text{sign}_{m'c^{(c)}_{m,n}} c^{(c)*}_{m',n'} \Pi^{(s,s)}_{m,n;m',n'} + \text{sign}_{m'c^{(c)}_{m,n}} c^{(c)*}_{m',n'} \Pi^{(s,s)}_{m,n;m',n'} \}
\]
The normalized modal acoustic impedance has been formulated as

\[
\left\{ \begin{array}{l}
\Pi^{(c,c)}_{m,m';n',n'} \\
\Pi^{(c,s)}_{m,m';n',n'} \\
\Pi^{(s,c)}_{m,m';n',n'} \\
\Pi^{(s,s)}_{m,m';n',n'}
\end{array} \right\} = \frac{k S^2}{8 \pi^2 \varrho_0 c} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta \left\{ \begin{array}{l}
M^{(c)}_{m,n}(k) M^{(c)*}_{m',n'}(k) \\
M^{(c)}_{m,n}(k) M^{(s)*}_{m',n'}(k) \\
M^{(s)}_{m,n}(k) M^{(c)*}_{m',n'}(k) \\
M^{(s)}_{m,n}(k) M^{(s)*}_{m',n'}(k)
\end{array} \right\} \frac{d\xi d\eta}{\gamma}
\] (18)

for \( m, m' \geq 0 \). The modal reference acoustic power must be a nonzero value and therefore it has been defined as

\[
\Pi_{m,m';n',n'}^{(\text{Ref.})} = \sqrt{\Pi_{m;n}^{(\text{Ref.})} \Pi_{m'\ell}^{(\text{Ref.})}} = \frac{1}{2} \varrho_0 c S \omega_{m,n} \omega_{m',n'}
\] (19)

where

\[
\Pi_{m;n}^{(\text{Ref.})} = \frac{1}{2} \varrho_0 c \int_{S} |\psi^{(c)}_{m,n}(r)|^2 dS = \frac{1}{2} \varrho_0 c \omega_{m,n}^2
\] (20)

The normalized modal acoustic impedance has been formulated as

\[
\left\{ \begin{array}{l}
\zeta^{(c,c)}_{m,m';n',n'} \\
\zeta^{(c,s)}_{m,m';n',n'} \\
\zeta^{(s,c)}_{m,m';n',n'} \\
\zeta^{(s,s)}_{m,m';n',n'}
\end{array} \right\} = \frac{1}{\Pi_{m,m';n',n'}^{(\text{Ref.})}} \left\{ \begin{array}{l}
\Pi^{(c,c)}_{m,m';n',n'} \\
\Pi^{(c,s)}_{m,m';n',n'} \\
\Pi^{(s,c)}_{m,m';n',n'} \\
\Pi^{(s,s)}_{m,m';n',n'}
\end{array} \right\}
\]

\[
= \frac{k S}{2 \pi} (-1)^{m+m'+p} \sqrt{\varepsilon_m \varepsilon_{m'}} \int_{0}^{\infty} d\tau \left\{ \begin{array}{l}
\phi^{(c,c)}_{m,m'} \\
\phi^{(c,s)}_{m,m'} \\
\phi^{(s,c)}_{m,m'} \\
\phi^{(s,s)}_{m,m'}
\end{array} \right\} \psi_{m,n}(\tau) \psi_{m',n'}(\tau) \frac{d\tau}{\gamma}
\] (21)

where the definite integral calculated along the angle coordinate \( \alpha \) of the wavevector assumes the form of

\[
\left\{ \begin{array}{l}
\phi^{(c,c)}_{m,m'} \\
\phi^{(c,s)}_{m,m'} \\
\phi^{(s,c)}_{m,m'} \\
\phi^{(s,s)}_{m,m'}
\end{array} \right\} = \frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{\cos m \alpha \cos m' \alpha}{\cos m \alpha \sin m' \alpha} \frac{\sin m \alpha \cos m' \alpha}{\sin m \alpha \sin m' \alpha} \, d\alpha = \frac{\delta_{m,m'}}{\varepsilon_m} \left\{ \begin{array}{l}
1 \\
0 \\
0 \\
\text{sign}_{m,m'}
\end{array} \right\}
\] (22)
Finally it has been obtained that \( \zeta^{(c,s)}_{m,n,m',n'} = \zeta^{(c,c)}_{m,n,m',n'} = 0 \) for any values of \( m \) and \( m' \), \( \zeta^{(s,s)}_{m,n,m',n'} = 0 \) for \( m \neq m' \) and

\[
\begin{pmatrix}
\zeta^{(c,c)}_{m,n,m',n'} \\
\zeta^{(s,s)}_{m,n,m',n'}
\end{pmatrix} = \frac{kS}{2\pi} \left\{ \frac{1}{\text{sign}} \right\} \int_{0}^{\infty} \psi_{m,n}(\tau)\psi_{m,n'}(\tau) \frac{\tau d\tau}{\gamma} \tag{23}
\]

Using Eqs. (18)–(23) has made it possible to express acoustic power from Eq. (17) in the form of

\[
\Pi = \sum_{m} \sum_{n,n'} \frac{\omega^2}{\omega_{m,n} \omega_{m,n'}} \left\{ c_{m,n}^{(c,c)} c_{m,n'}^{(c,c)} + \text{sign}_{m,n} c_{m,n}^{(s,s)} c_{m,n'}^{(s,s)} \right\} \Pi_{m,n,n'}
\]

\[
= \frac{1}{2} \varrho c \omega^2 S \sum_{m} \sum_{n,n'} \left\{ c_{m,n}^{(c,c)} c_{m,n'}^{(c,c)} + \text{sign}_{m,n} c_{m,n}^{(s,s)} c_{m,n'}^{(s,s)} \right\} \zeta_{m,n,n'} \tag{24}
\]

where it has been denoted

\[
\Pi_{m,n,n'} \equiv \Pi^{(c,c)}_{m,n,m',n'} = \frac{kS^2}{8\pi^2 \varrho c} \int_{-\infty}^{+\infty} \frac{M_{m,n}(k) M_{m,n'}^{(c,c)}(k)}{k} d\xi d\eta \tag{25a}
\]

\[
\zeta_{m,n,n'} \equiv \zeta^{(c,c)}_{m,n,m',n'} = \frac{kS}{2\pi} \int_{0}^{\infty} \psi_{m,n}(\tau) \psi_{m,n'}(\tau) \frac{\tau d\tau}{\gamma} \tag{25b}
\]

Considering Eqs. (14) and (15) it is easy to note that

\[
\Pi_{m,n,n'} = \Pi_{m,n',n}; \quad \zeta_{m,n,n'} = \zeta_{m,n',n} \tag{26}
\]

which can be useful for reducing the time necessary for numerical computations of the mutual impedance.

The reference acoustic power necessary to normalize acoustic power from Eq. (24) has been defined similarly as in Eq. (20) using orthogonality condition from Eq. (6)

\[
\Pi^{(\text{Ref})} = \frac{1}{2} \varrho c S \langle |v_N|^2 \rangle \tag{27}
\]

where the mean square vibration velocity has been defined as

\[
\langle |v_N|^2 \rangle = \frac{1}{S} \int_{S} |v_N(r)|^2 dS = \omega^2 \sum_{m} \sum_{n} \left\{ |c_{m,n}^{(c,c)}|^2 + \text{sign}_{m,n} |c_{m,n}^{(s,s)}|^2 \right\} \tag{28}
\]

The normalized acoustic impedance has been formulated similarly as in Eq. (21)

\[
\zeta = \frac{\Pi}{\Pi^{(\text{Ref})}} = \frac{\omega^2}{\langle |v_N|^2 \rangle} \sum_{m,n,n'} \left\{ c_{m,n}^{(c,c)} c_{m,n'}^{(c,c)} + \text{sign}_{m,n} c_{m,n}^{(s,s)} c_{m,n'}^{(s,s)} \right\} \zeta_{m,n,n'} \tag{29}
\]

The equation of motion of the excited membrane is

\[
(k_T^{-2} \nabla^2 + 1) W(r, \varphi) = -\frac{1}{\omega^2 \sigma} \{ f(r, \varphi) + p(r, \varphi, 0) \} \tag{30}
\]
where \( k_T = \omega / c_M \) is the structural wavenumber, \( \omega \) is the excitation circular frequency, \( f \) is the time-harmonic excitation, and \( p \) is the acoustic pressure from the air column. So, the term \( p \) represents the acoustic attenuation from the surrounding gaseous medium located above the vibrating membrane with the assumption that there is vacuum below it. If it is assumed that the gaseous medium appears on both sides of the membrane then the sound pressure should be taken twice in the equation above. Equation (30) has further been expressed as the following algebraic equation system

\[
\left\{ \begin{array}{c}
c_{m,n}^{(c)} \\
c_{m,n}^{(s)}
\end{array} \right\} \left( \frac{k_{m,n}^2}{k_T^2} - 1 \right) = \frac{1}{\omega^2 \sigma} \left( \left\{ \begin{array}{c} f_{m,n}^{(c)} \\
f_{m,n}^{(s)}
\end{array} \right\} + \left\{ \begin{array}{c} p_{m,n}^{(c)} \\
p_{m,n}^{(s)}
\end{array} \right\} \right)
\]

(31)

after applying Eq. (7) together with orthogonality condition satisfied by the trigonometric functions where it has been denoted

\[
\left\{ \begin{array}{c}
f_{m,n}^{(c)} \\
f_{m,n}^{(s)}
\end{array} \right\} = \frac{1}{S} \int_S \left\{ \begin{array}{c} W_{m,n}^{(c)}(r, \varphi) \\
W_{m,n}^{(s)}(r, \varphi)
\end{array} \right\} f(r, \varphi)dS
\]

(32a)

\[
\left\{ \begin{array}{c}
p_{m,n}^{(c)} \\
p_{m,n}^{(s)}
\end{array} \right\} = \frac{1}{S} \int_S \left\{ \begin{array}{c} W_{m,n}^{(c)}(r, \varphi) \\
W_{m,n}^{(s)}(r, \varphi)
\end{array} \right\} p(r, \varphi, 0)dS
\]

(32b)

As mentioned above the sound pressure represents the acoustic attenuation and it is necessary to insert the factors

\[
\left\{ \begin{array}{c}
p_{m,n}^{(c)} \\
p_{m,n}^{(s)}
\end{array} \right\} = -i \omega \rho_0 c \sum_{n'} \zeta_{m;n',n} \left\{ \begin{array}{c} c_{m,n'}^{(c)} \\
c_{m,n'}^{(s)}
\end{array} \right\}
\]

(33)

to Eq. (31) which gives

\[
\left\{ \begin{array}{c}
c_{m,n}^{(c)} \\
c_{m,n}^{(s)}
\end{array} \right\} \left( \frac{k_{m,n}^2}{k_T^2} - 1 \right) + i \varepsilon_0 \sum_{n'} \zeta_{m;n',n} \left\{ \begin{array}{c} c_{m,n'}^{(c)} \\
c_{m,n'}^{(s)}
\end{array} \right\} = \frac{1}{\omega^2 \sigma} \left\{ \begin{array}{c} f_{m,n}^{(c)} \\
f_{m,n}^{(s)}
\end{array} \right\}
\]

(34)

where the dimensionless acoustic attenuation factor is

\[
\varepsilon_0 = \frac{\rho_0}{\sigma k}
\]

(35)

If the factor \( \varepsilon_0 \) is small enough (small fluid loading) all the terms representing acoustic attenuation in Eq. (34) can be neglected. This is true when \( k_T \) is much different than \( k_{m,n} \), i.e. when the coupling factors \( c_{m,n} \) and the membrane’s transverse deflection assume some finite values. This gives briefly

\[
\left\{ \begin{array}{c}
c_{m,n}^{(c)} \\
c_{m,n}^{(s)}
\end{array} \right\} \approx \frac{1}{\omega^2 \sigma \left( k_T^{-2} k_{m,n}^2 - 1 \right)} \left\{ \begin{array}{c} f_{m,n}^{(c)} \\
f_{m,n}^{(s)}
\end{array} \right\}
\]

(36)
with no necessity to solve the algebraic equation system. However, for $k_T$ close enough to
the discrete values of $k_{m,n}$ the vibration resonance appears and the acoustic attenuation
plays an essential role to limit the membrane’s deflection. In this situation the attenuation
has to be included even for small fluid loading.

Solving equation system (34) yields the values of the components $c^{(c)}_{m,n}$ and $c^{(s)}_{m,n}$ of the
coupling matrix of acoustic power and acoustic impedance with their modal counterparts
from Eqs. (24) and (29). Further, applying the coupling matrix has made it possible to
rearrange Eq. (34) into the form of

$$\sum_{n'} \zeta_{m;n,n'} \left\{ \begin{array}{l} c^{(c)}_{m,n} \\ c^{(s)}_{m,n} \end{array} \right\} = \frac{i}{\varepsilon_0} \left( \frac{k_{m,n}^2}{k_T^2} - 1 \right) \left( \begin{array}{c} c^{(c)}_{m,n} \\ c^{(s)}_{m,n} \end{array} \right) - \frac{1}{\omega^2 \sigma} \left( f^{(c)}_{m,n} \right)$$

and after inserting into Eqs. (24) and (29) it has been obtained that

$$\Pi = \frac{i}{2\varepsilon_0} \sigma_0 \omega^2 S \sum_{m,n} \left( c^{(c)*}_{m,n} c^{(c)}_{m,n} \left( \frac{k_{m,n}^2}{k_T^2} - 1 \right) - \frac{f^{(c)}_{m,n}}{\omega^2 \sigma} \right)$$

$$+ \text{sign}_{m,n} e^{(s)*}_{m,n} e^{(s)}_{m,n} \left( \frac{k_{m,n}^2}{k_T^2} - 1 \right) - \frac{f^{(s)}_{m,n}}{\omega^2 \sigma} \right)$$

$$\zeta = \frac{i\omega^2}{\varepsilon_0 (|v|^2)} \sum_{m,n} \left( c^{(c)*}_{m,n} c^{(c)}_{m,n} \left( \frac{k_{m,n}^2}{k_T^2} - 1 \right) - \frac{f^{(c)}_{m,n}}{\omega^2 \sigma} \right)$$

$$+ \text{sign}_{m,n} e^{(s)*}_{m,n} e^{(s)}_{m,n} \left( \frac{k_{m,n}^2}{k_T^2} - 1 \right) - \frac{f^{(s)}_{m,n}}{\omega^2 \sigma} \right)$$

It is necessary to remember that Eq. (33) cannot be used in Eqs. (38) where the coupling
matrix must be obtained by solving equation system given in Eq. (34). In other words,
Eqs. (38) can only be used when the acoustic attenuation is included. Additionally, it has
been maintained before Eq. (2) that the sound pressure is time harmonic. The consequence
of that assumption is that the acoustic power and the modal impedance presented in this
paper are valid only for a single frequency.

3. Axisymmetric Asymptotics

After inserting Eq. (15) into Eq. (25b) the normalized modal radiation impedance has been
formulated as

$$\zeta_{m;n,n'} = \theta_{m;n,n'} - i\chi_{m;n,n'}$$

where

$$\theta_{m;n,n'} = \frac{kS}{2\pi} \int_0^k \psi_{m,n}(\tau) \psi_{m,n'}(\tau) \frac{\tau d\tau}{\sqrt{k^2 - \tau^2}}$$

$$= 2\delta_{m,n} \delta_{m,n'} \int_0^1 \frac{J_m^2(\beta x) dx}{(x^2 - \delta_{m,n}^2)(x^2 - \delta_{m,n'}^2)\sqrt{1 - x^2}}$$

(40a)
\[
\chi_{m;n,n'} = \frac{kS}{2\pi} \int_k^\infty \psi_{m,n}(\tau)\psi_{m,n'}(\tau)\frac{\tau d\tau}{\sqrt{\tau^2 - k^2}}
= 2\delta_{m,n}\delta_{m,n'} \int_1^\infty \frac{J_m^2(\beta x)dx}{(x^2 - \delta_{m,n}^2)(x^2 - \delta_{m,n'}^2)\sqrt{x^2 - 1}}
\]

(40b)

are the normalized modal radiation resistance and reactance, respectively, \(\beta = k\alpha\) is the
dimensionless wavenumber, \(\delta_{m,n} = \beta_{m,n}/\beta = k_{m,n}/k\) is the normalized structural wave-
umber, and the following substitution \(\tau = kx\) has been applied.

3.1. Axisymmetric modal self-impedance. High frequencies \(\delta_n \equiv \delta_{0,n} < 1\)

The normalized modal radiation self-resistance of the axisymmetric mode \((0,n)\) has been
obtained from Eq. (40a) assuming that \(n = n'\) which has given

\[
\theta_n = \theta_{0;n,n} = 2\delta_n^2 \int_0^1 \frac{J_m^2(\beta x)dx}{(x^2 - \delta_n^2)^2\sqrt{1 - x^2}}
\]

(41)

where \(\delta_n \equiv \delta_{0,n} = \beta_{n}/\beta = k_n/k\). The asymptotic formulation of this quantity was reported earlier.\(^{32}\) It was obtained using the methods of contour integral analysis and stationary
phase.\(^{26,33}\) For this purpose the following contour integral was analyzed

\[
\int_C \frac{zF(z)dz}{\sqrt{1 - z^2(z^2 - \delta_n^2)^2}} = 0
\]

(42)

where

\[
F(z) = J_0(\beta z)H_0^{(1)}(\beta z)
\]

(43)

with the integration contour presented in Fig. 1. This made it possible to formulate the radiation resistance from Eq. (41) in the form of

\[
\theta_n = 2\delta_n^2 \Re \left\{ \pi i F'(\delta_n) \right\} + 2\delta_n^2 \int_1^\infty \frac{\Im\{F(z)\}dx}{(x^2 - \delta_n^2)^2\sqrt{x^2 - 1}}
\]

(44)

where

\[
F(z) = \frac{zF(z)}{\sqrt{1 - z^2(z^2 + \delta_n)^2}}
\]

(45)

The residuum contribution in Eq. (44) was calculated exactly whereas the contribution from
the definite integral was calculated using the zero term of the corresponding asymptotic
expansion

\[
\Im\{F(z)\} = J_0(\beta x)Y_0(\beta x) \approx -\frac{\cos(2\beta x)}{\pi \beta x}
\]

(46)

and the stationary phase method giving

\[
\theta_n \approx \frac{1}{\sqrt{1 - \delta_n^2}} - \frac{\delta_n^2 \cos(2\beta + \pi/4)}{\beta \sqrt{\pi \beta (1 - \delta_n^2)^2}}
\]

(47)

which is valid for \(\beta > \beta_n\) and the corresponding approximation error order is \(\beta^{-3/2}\delta_n^4\).
So far, there is no asymptotic formulation reported in the literature for the radiation self-reactance of the axisymmetric mode of the vibrating circular membrane. However, it has been given for a vibrating clamped circular plate in Ref. 33. In this paper the zero term of asymptotic expansion\(^{38}\)

\[ J_0^2(\beta x) \approx \frac{1}{\pi \beta x}(1 - \sin 2\beta x) \] (48)

has been used together with the stationary phase method in a similar way

\[ \chi_n \equiv \chi_{0;n,n} = 2\delta_n^2 \int_1^\infty \frac{J_0^2(\beta x)dx}{(x^2 - \delta_n^2)^2 \sqrt{x^2 - 1}} = \bar{I} + \tilde{I} \] (49)

where the corresponding integrals have been denoted as

\[ \bar{I} = \frac{2\delta_n^2}{\pi \beta} \int_1^\infty \frac{dx}{(x^2 - \delta_n^2)^2 \sqrt{x^2 - 1}} \approx \frac{1}{\pi \beta(1 - \delta_n^2)} - \frac{(1 - 2\delta_n^2)\arcsin \delta_n}{\pi \beta \delta_n(1 - \delta_n^2)^{3/2}} \] (50a)

\[ \tilde{I} = \frac{2\delta_n^2}{\pi \beta} \int_1^\infty \frac{\sin(2\beta x)dx}{(x^2 - \delta_n^2)^2 \sqrt{x^2 - 1}} \approx \frac{\delta_n^2 \sin(2\beta + \pi/4)}{\beta \sqrt{\pi \beta(1 - \delta_n^2)^2}} \] (50b)

with the approximation error order as in Eq. (47).

3.2. Axisymmetric modal self-impedance. Low frequencies \(\delta_n > 1\)

The question has arisen whether it is possible to obtain asymptotic formulas valid for the circular membrane and below the coincidence, i.e. for \(7 < \beta < \beta_n\). Similar formulas have been presented earlier for the acoustic reactance of an elastically supported circular plate\(^{34}\) and the same method has been used in this paper for both radiation resistance and radiation reactance. For this purpose, the phase change of the term \(\sqrt{1 - \delta_n^2}\) has been used at the
transition from the frequency band $\beta_n < \beta$ to $\beta < \beta_n$, together with the branch cut in the plane of complex variable of the function $\arcsin \delta_n$ which results in the relation

$$\arcsin \delta_n = \frac{\pi}{2} - i \arccosh \delta_n$$

for $\delta_n > 1$ given that $\delta_n \in \mathbb{R}$. As a result it has been obtained that

$$\frac{\arcsin \delta_n}{\sqrt{1 - \delta_n^2}} = -\frac{1}{\sqrt{\delta_n^2 - 1}} \left( \arccosh \delta_n + \frac{\pi i}{2} \right)$$

(51)

and

$$\theta_n \approx -\frac{1 - 2\delta_n^2}{2\beta \delta_n (\delta_n^2 - 1)^{3/2}} - \frac{\delta_n^2 \cos(2\beta + \pi/4)}{\beta \sqrt{\pi \beta (1 - \delta_n^2)^2}}$$

(53a)

$$\chi_n \approx \frac{1}{\sqrt{\delta_n^2 - 1}} + \frac{1}{\pi \beta (1 - \delta_n^2)} \left( \frac{1 - 2\delta_n^2}{\beta \sqrt{\pi \beta (1 - \delta_n^2)^2}} \right)$$

(53b)

which corresponds to the frequency band $7 < \beta < \beta_n$. Equations (53) represent wide frequency band asymptotic formulas for $\beta \in (7, \beta_n)$ introducing the approximation error order $\beta^{-3/2} \delta_n^{-4} \leq 10^{-3}$. It is worth noticing that the terms containing the cosine and sine functions in the above equations are identical as in Eqs. (47) and (50b) which is obvious since these terms do not change their phase as their frequency varies from that above the coincidence to that below it.

3.3. Axisymmetric modal mutual impedance. High frequencies

$\delta_n < \delta_{n'} < 1$

In this case, the calculations have been conducted differently than in the earlier paper. The following identity has been used

$$\frac{1}{(x^2 - \delta_n^2)(x^2 - \delta_{n'}^2)} = \frac{1}{\delta_n^2 - \delta_{n'}^2} \left( \frac{1}{x^2 - \delta_n^2} - \frac{1}{x^2 - \delta_{n'}^2} \right)$$

(54)

assuming that $\delta_n < \delta_{n'}$ and $\beta_n < \beta_{n'}$. Equations (40) have been formulated as sums consisting of two terms, each for $m = 0$

$$\theta_{n,n'} = \theta_{0;n,n'} = \theta_{n,n'} - \theta_{n,(n')}$$

(55a)

$$\chi_{n,n'} = \chi_{0;n,n'} = \chi_{n,n'} - \chi_{n,(n')}$$

(55b)

where the parentheses in indices indicate the mode of the corresponding pole

$$\theta_{n,n'} = \frac{2\delta_n \delta_{n'}}{\delta_n^2 - \delta_{n'}^2} \int_0^1 \frac{J^2_\theta(\beta x) dx}{(x^2 - \delta_n^2) \sqrt{1 - x^2}}, \quad \theta_{n,(n')} = \frac{2\delta_n \delta_{n'}}{\delta_n^2 - \delta_{n'}^2} \int_0^1 \frac{J^2_\theta(\beta x) dx}{(x^2 - \delta_n^2) \sqrt{1 - x^2}}$$

(56a)

$$\chi_{n,n'} = \frac{2\delta_n \delta_{n'}}{\delta_n^2 - \delta_{n'}^2} \int_1^\infty \frac{J^2_\theta(\beta x) dx}{(x^2 - \delta_n^2) \sqrt{x^2 - 1}}, \quad \chi_{n,(n')} = \frac{2\delta_n \delta_{n'}}{\delta_n^2 - \delta_{n'}^2} \int_1^\infty \frac{J^2_\theta(\beta x) dx}{(x^2 - \delta_n^2) \sqrt{x^2 - 1}}$$

(56b)
The integral in Eq. (56a) has been calculated first. The Cauchy theorem about the contour integral has been used to give (cf. Fig. 1)

$$\oint_C \frac{zF(z)}{(z^2 - \delta_n^2)\sqrt{1 - z^2}} \, dz = 0$$

(57)

where the contour $C$ is the Jordan curve and the function $F(z)$ given by Eq. (43) is analytic on contour $C$ and in its interior. Further Eq. (57) has been formulated as

$$\int_0^1 x \text{Re}\{F(x)\} \, dx + \int_1^\infty x \text{Im}\{F(x)\} \, dx = \text{Re}\{\pi i F(\delta_n)\}$$

(58)

where

$$F(z) = \frac{zF(z)}{\sqrt{1 - z^2}(z + \delta_n)}$$

(59)

The fact that the integral calculated on the large arc of contour $C$ approaches zero for $R \to \infty$ has been deduced from the Jordan’s lemma. Similarly, the residues in the singular points $z = 0$ and $z = 1$ are equal to zero. Since $F(iy) = (-2i/\pi)J_0(\beta y)K_0(\beta y)$, $\text{Re} \, F(iy) \equiv 0$ and the integral calculated over variable $y$ within the limits from $+\infty$ to $0$ is also equal to zero. Moreover, because of the frequency equation $J_m(\beta_m) = 0$ it has been obtained that

$$\text{Re}\{\pi i F(\delta_n)\} = -\pi J_\lambda(\lambda_n)Y_0(\lambda_n)$$

(60)

So Eq. (58) has been formulated in the form of

$$\int_0^1 J_0^2(\beta x) \, dx = \int_1^\infty J_0(\beta x) Y_0(\beta x) \, dx$$

(61)

and after including the zero term of the asymptotic expansion of product from Eq. (46) the value of integral in Eq. (61) has been formulated as

$$\frac{1}{\pi \beta} \int_1^\infty \frac{\cos(2\beta x) \, dx}{(x^2 - \delta_n^2)\sqrt{x^2 - 1}}$$

(62)

Further the stationary phase method has been applied

$$U(\beta) = \int_1^\infty \frac{\exp(2i\beta x) \, dx}{x^2(\beta^2 - \delta_n^2)\sqrt{x^2 - 1}}$$

(63)

It has been assumed that $r = 0$, $S(x) = x$ and the following substitution has been used $t = \sqrt{x^2 - 1}$ yielding

$$U_0(\beta, t_0) = \int_{t_0 - \epsilon}^{t_0 + \epsilon} f(t) \exp\{i\beta S(t)\} \, dt$$

$$\approx \sqrt{\frac{2\pi}{\beta |S''(t_0)|}} \left[ f(t_0) + O\left(\frac{1}{\beta}\right) \right] \exp\left(i\left\{\beta S(t_0) + \frac{\pi}{4} \text{sign}[S''(t_0)]\right\}\right)$$

(64)
for \( \epsilon \to 0 \). The asymptotic value of the integral from Eq. (63) has been obtained

\[
U_0(\beta, 0) = \frac{1}{2} \int_0^\epsilon f(t) \exp\{i\beta S(t)\} dt \\
\approx \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \left\{ \frac{1}{1 - \delta_n^2} + O\left(\frac{1}{\beta}\right) \right\} \exp\{i(2\beta + \pi/4)\} 
\]

(65)

After inserting into Eq. (62)

\[
\frac{1}{2\beta\sqrt{\pi\beta}} \left\{ \frac{1}{1 - \delta_n^2} + O\left(\frac{1}{\beta}\right) \right\} \cos(2\beta + \pi/4) 
\]

(66)

After using this value in Eq. (61) and further in Eq. (56a) it has been obtained that

\[
\theta(n,n') \approx -\delta_n \delta_{n'} \cos(2\beta + \pi/4) \frac{\delta_n^2 - \delta_{n'}^2}{\beta \sqrt{\pi\beta}(1 - \delta_n^2)} , \quad \theta(n',n) \approx -\delta_n \delta_{n'} \cos(2\beta + \pi/4) \frac{\delta_n^2 - \delta_{n'}^2}{\beta \sqrt{\pi\beta}(1 - \delta_n^2)} 
\]

(67)

After inserting into Eq. (55a) it assumes the form of

\[
\theta(n,n') \approx -\delta_n \delta_{n'} \cos(2\beta + \pi/4) \frac{\delta_n^2 - \delta_{n'}^2}{\beta \sqrt{\pi\beta}(1 - \delta_n^2)} (1 - \delta_{n'}^2) 
\]

(68)

with the approximation error smaller than \( \beta^{-3/2}\delta_n^2\delta_{n'}^2 \). The values of integrals from Eqs. (56b) have been calculated similarly as the former one using directly the stationary phase method

\[
\chi(n,n') \approx \frac{\delta_n \delta_{n'}}{\delta_n^2 - \delta_{n'}^2} \left\{ \frac{2 \arcsin \delta_n}{\pi \beta \delta_n \sqrt{1 - \delta_n^2}} + \frac{\sin(2\beta + \pi/4)}{\beta \sqrt{\pi\beta}(1 - \delta_n^2)} \right\} 
\]

(69a)

\[
\chi(n',n) \approx \frac{\delta_n \delta_{n'}}{\delta_n^2 - \delta_{n'}^2} \left\{ \frac{2 \arcsin \delta_{n'}}{\pi \beta \delta_{n'} \sqrt{1 - \delta_{n'}^2}} + \frac{\sin(2\beta + \pi/4)}{\beta \sqrt{\pi\beta}(1 - \delta_{n'}^2)} \right\} 
\]

(69b)

After inserting into Eq. (55b) it has been obtained that

\[
\chi(n,n') = \frac{2}{\pi \beta (\delta_n^2 - \delta_{n'}^2)} \left( \frac{\delta_{n'} \arcsin \delta_n}{\sqrt{1 - \delta_n^2}} - \frac{\delta_n \arcsin \delta_{n'}}{\sqrt{1 - \delta_{n'}^2}} \right) + \frac{\delta_n \delta_{n'} \sin(2\beta + \pi/4)}{\beta \sqrt{\pi\beta}(1 - \delta_n^2)(1 - \delta_{n'}^2)} 
\]

(70)

with the approximation error order as in Eq. (68).

3.4. Axisymmetric modal mutual impedance. Low frequencies \( 1 < \delta_n < \delta_{n'} \)

In this case, the corresponding asymptotic formulas have been obtained directly from Eqs. (68), (70), (39), (51) and (52)

\[
\theta(n,n') \approx -\frac{1}{\beta (\delta_n^2 - \delta_{n'}^2)} \left( \frac{\delta_{n'}}{\sqrt{\delta_n^2 - 1}} - \frac{\delta_n}{\sqrt{\delta_{n'}^2 - 1}} \right) - \frac{\delta_n \delta_{n'} \cos(2\beta + \pi/4)}{\beta \sqrt{\pi\beta}(1 - \delta_n^2)(1 - \delta_{n'}^2)} 
\]

(71a)
Further Eqs. (67) and (72a) have been summed up as well as Eqs. (69a) and (72b) yielding

\[
\chi_{n,n'} \approx \frac{-2}{\pi \beta (\delta_n^2 - \delta_{n'}^2)} \left( \frac{\delta_n \arccosh \delta_n}{\sqrt{\delta_n^2 - 1}} - \frac{\delta_n \arccosh \delta_{n'}}{\sqrt{\delta_{n'}^2 - 1}} \right) + \frac{\delta_n \delta_{n'} \sin(2 \beta + \pi/4)}{\beta \sqrt{\pi \beta (1 - \delta_n^2)(1 - \delta_{n'}^2)}}
\]  

(71b)

with the approximation error smaller than \( \beta^{-3/2}\delta_n^{-2}\delta_{n'}^{-2} \).

3.5. **Axisymmetric modal mutual impedance. Middle frequencies**

\( \delta_n < 1 < \delta_{n'} \)

In this case, the calculations have been performed a little differently than in the case the former frequency bands. The corresponding formulas from Eqs. (67) and (69b) have been rearranged to their low frequency counterparts using Eqs. (51) and (52)

\[
\theta_{n,(n')} \approx \frac{1}{\beta (\delta_n^2 - \delta_{n'}^2)} \left\{ \frac{1}{\delta_{n'} \sqrt{\delta_{n'}^2 - 1}} + \frac{\cos(2 \beta + \pi/4)}{\sqrt{\pi \beta (1 - \delta_{n'}^2)}} \right\}
\]  

(72a)

\[
\chi_{n,(n')} \approx \frac{-1}{\beta (\delta_n^2 - \delta_{n'}^2)} \left\{ 2 \arccosh \delta_{n'} \sqrt{\delta_{n'}^2 - 1} - \frac{\sin(2 \beta + \pi/4)}{\sqrt{\pi \beta (1 - \delta_{n'}^2)}} \right\}
\]  

(72b)

Further Eqs. (67) and (72a) have been summed up as well as Eqs. (69a) and (72b) yielding

\[
\theta_{n,n'} \approx \frac{\delta_n}{\beta (\delta_n^2 - \delta_{n'}^2) \sqrt{\delta_{n'}^2 - 1}} + \frac{\delta_n \delta_{n'} \cos(2 \beta + \pi/4)}{\beta \sqrt{\pi \beta (1 - \delta_n^2)(1 - \delta_{n'}^2)}}
\]  

(73a)

\[
\chi_{n,n'} \approx \frac{2}{\pi \beta (\delta_n^2 - \delta_{n'}^2)} \left( \frac{\delta_{n'} \arcsin \delta_n}{\sqrt{1 - \delta_n^2}} + \frac{\delta_n \arccosh \delta_{n'}}{\sqrt{\delta_{n'}^2 - 1}} \right) + \frac{\delta_n \delta_{n'} \sin(2 \beta + \pi/4)}{\beta \sqrt{\pi \beta (1 - \delta_n^2)(1 - \delta_{n'}^2)}}
\]  

(73b)

with the approximation error smaller than \( \beta^{-3/2}\delta_n^{-2}\delta_{n'}^{-2} \). It is worth noticing that the pole for \( z = \delta_n \) contributes the modal acoustic impedance in its high frequency band while the pole for \( z = \delta_{n'} \) contributes the impedance in its low frequency band. Obviously, this is also reflected in the corresponding approximation error order in Eq. (73).

4. **Asymmetric Asymptotics**

In the case when the membrane vibrates asymmetrically \( m \geq 1 \) then the radiated waves are also asymmetric. It is therefore necessary to adapt the formulas of the previous section so that they are useful for \( m \geq 1 \). However, the use of the zero asymptotic expansion term only for the product of special functions such as in Eqs. (46) and (48) is not sufficient for this purpose, since with the increase in \( m \), the estimate approximation error value increases
and significantly exceeds the theoretical error value. Therefore using some more asymptotic expansion terms has become useful\textsuperscript{38,39}

\[ J_m(z) \approx \sqrt{\frac{2}{\pi z}} \left\{ P_m(z) \cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) - Q_m(z) \sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \right\} \quad (74a) \]

\[ Y_m(z) \approx \sqrt{\frac{2}{\pi z}} \left\{ P_m(z) \sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) + Q_m(z) \cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \right\} \quad (74b) \]

for \( z \gg 1 \) and \( m \geq 1 \) where it has been denoted

\[ P_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!(8z)^{2k}} \prod_{r=0}^{2k-1} \{(2m)^2 - (2r + 1)^2\} \quad (75a) \]

\[ Q_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(8z)^{2k+1}} \prod_{r=0}^{2k} \{(2m)^2 - (2r + 1)^2\} \quad (75b) \]

The expressions \( P_m(z) \) and \( Q_m(z) \) include a number of terms \( N \), the value of which will be determined later. Further the asymptotic expansions of the products have been formulated as

\[ J_m(z)Y_m(z) \approx \frac{(-1)^m}{\pi z} \left\{ \left(Q_m^2(z) - P_m^2(z)\right) \cos 2z + 2Q_m(z)P_m(z) \sin 2z \right\} \quad (76a) \]

\[ J_m^2(z) \approx \frac{1}{\pi z} \left\{ P_m^2(z) + Q_m^2(z) + (-1)^m \left[ \left(P_m^2(z) - Q_m^2(z)\right) \sin 2z + 2P_m(z)Q_m(z) \cos 2z \right] \right\} \quad (76b) \]

as well as the following values

\[ P_m^2(z) \approx \sum_{k,l=0}^{N} \frac{(-1)^{k+l} \epsilon_{m,2k-1}\epsilon_{m,2l-1}}{(8z)^{2k+2l}} \quad (77a) \]

\[ Q_m^2(z) \approx \sum_{k,l=0}^{N} \frac{(-1)^{k+l} \epsilon_{m,2k}\epsilon_{m,2l}}{(8z)^{2k+2l+2}} \]

\[ P_m(z)Q_m(z) \approx \sum_{k,l=0}^{N} \frac{(-1)^{k+l} \epsilon_{m,2k-1}\epsilon_{m,2l}}{(8z)^{2k+2l+1}} \quad (77b) \]

where it has been denoted

\[ \epsilon_{m,k} = \prod_{r=0}^{k} \frac{(2m)^2 - (2r + 1)^2}{r+1} \quad (78) \]

for \( k = 0, \ldots, N \).

4.1. \textit{Asymmetric modal self-impedance. High frequencies} \( \delta_{m,n} < 1 \)

The same procedure has been conducted as in the axisymmetric case. The method of the contour integral together with the corresponding frequency equation have been used for the
analysis of expressions of radiation resistance and radiation reactance given in Eqs. (40) for
\( n = n'\). The following formulas have been obtained instead of Eqs. (41), (44) and (49)
\[
\theta_{m,n} = \theta_{m,n,n} = \frac{1}{\sqrt{1 - \delta_{m,n}^2}} + 2\delta_{m,n}^2 \int_1^\infty \frac{J_m(\beta x)Y_m(\beta x)x \, dx}{(x^2 - \delta_{m,n}^2)^2 \sqrt{x^2 - 1}} \tag{79a}
\]
\[
\chi_{m,n} = \chi_{m,n,n} = 2\delta_{m,n}^2 \int_1^\infty \frac{J_m^2(\beta x)x \, dx}{(x^2 - \delta_{m,n}^2)^2 \sqrt{x^2 - 1}} \tag{79b}
\]
The expansions (76) have been inserted and the stationary phase method has been applied
separately for each term of the received series of integrals
\[
\int_1^\infty \frac{\exp(i2\beta x)dx}{x^\alpha(x^2 - \delta_{m,n}^2)^2 \sqrt{x^2 - 1}} \approx \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \left\{ 1 + O\left(\frac{1}{2\beta}\right) \right\} \exp\left\{ i(2\beta + \pi/4) \right\} \tag{80}
\]
The asymptotic expansion from Eq. (76b) has been used while calculating the acoustic
reactance. It has been therefore necessary to calculate the following value of integral
\[
I_k = \delta_{m,n}^2 \int_1^\infty \frac{dx}{x^{2k}(x^2 - \delta_{m,n}^2)^2 \sqrt{x^2 - 1}} \tag{81}
\]
for some few initial values of \( k = 0, 1, 2, \ldots \) and \( 0 < \delta_{m,n} < 1 \)
\[
I_0 = \frac{1}{2(1 - \delta_{m,n}^2)} - \frac{(1 - 2\delta_{m,n}^2) \arcsin \delta_{m,n}}{2\delta_{m,n}(1 - \delta_{m,n}^2)^{3/2}} \tag{82a}
\]
\[
I_1 = \frac{3 - 2\delta_{m,n}^2}{2\delta_{m,n}^2(1 - \delta_{m,n})} - \frac{(3 - 4\delta_{m,n}^2) \arcsin \delta_{m,n}}{2\delta_{m,n}^3(1 - \delta_{m,n}^2)^{3/2}} \tag{82b}
\]
\[
I_2 = \frac{3 + 2\delta_{m,n}^2}{6\delta_{m,n}} - \frac{\arcsin \delta_{m,n}}{2\delta_{m,n}^3 \sqrt{1 - \delta_{m,n}^2}} \tag{82c}
\]
The value of the integral for \( k \geq 3 \) has been formulated as follows \(\text{cf. Eq. (50a)}\)
\[
I_k = \frac{1}{\delta_{m,n}^2} \left\{ I_{k-1} + \frac{\sqrt{\pi} \Gamma(k)}{4\Gamma(k + 1/2)} \right\} \tag{83}
\]
after using mathematical induction. Further, the normalized radiation resistance and
reactance have been formulated in the form of
\[
\theta_{m,n} \approx \frac{1}{\sqrt{1 - \delta_{m,n}^2}} + \chi_{m,n} \cos(2\beta + \pi/4) + \gamma_{m,n} \sin(2\beta + \pi/4) \tag{84a}
\]
\[
\chi_{m,n} \approx \mathcal{U} - \frac{\nu \arcsin \delta_{m,n}}{2\delta_{m,n} \sqrt{1 - \delta_{m,n}^2}} + \gamma_{m,n} \cos(2\beta + \pi/4) - \chi_{m,n} \sin(2\beta + \pi/4) \tag{84b}
\]
where it has been denoted
\[ x_{m,n} = \frac{(-1)^{m} \delta_{m,n}^{2} \{ Q_{m}^{2}(\beta) - P_{m}^{2}(\beta) \} \beta \sqrt{\pi \beta (1 - \delta_{m,n}^{2})}}{2 \sqrt{\pi} \beta (1 - \delta_{m,n}^{2})^2}, \quad y_{m,n} = \frac{(-1)^{m} \delta_{m,n}^{2} 2 P_{m}(\beta) Q_{m}(\beta) \beta \sqrt{\pi \beta (1 - \delta_{m,n}^{2})}}{2 \sqrt{\pi} \beta (1 - \delta_{m,n}^{2})^2} \] (85a)

\[ u \approx \frac{2}{\pi \beta} \sum_{k,l=0}^{N} (-1)^{k+l} \left\{ \frac{\epsilon_{m,2k} \epsilon_{m,2l} w_{k+l}(\delta_{m,n})}{(8 \beta)^{2k+2l+2}} + \frac{\epsilon_{m,2k-1} \epsilon_{m,2l-1} w_{k+l}(\delta_{m,n})}{(8 \beta)^{2k+2l}} \right\} \] (85b)

\[ v \approx \frac{2}{\pi \beta} \sum_{k,l=0}^{N} (-1)^{k+l} \left\{ \frac{\epsilon_{m,2k} \epsilon_{m,2l} w_{k+l}(\delta_{m,n})}{(8 \beta)^{2k+2l+2}} + \frac{\epsilon_{m,2k-1} \epsilon_{m,2l-1} w_{k+l}(\delta_{m,n})}{(8 \beta)^{2k+2l}} \right\} \] (85c)

\[ w_{k}(\delta_{m,n}) = \begin{cases} \frac{1}{2(1 - \delta_{m,n}^{2})^{2}} & k = 0 \\ 3 - 2 \delta_{m,n}^{2} w_{0}(\delta_{m,n}) & k = 1 \\ \frac{3 + 2 \delta_{m,n}^{2}}{6 \delta_{m,n}^{4}} & k = 2 \\ \frac{1}{\delta_{m,n}^{2}} \left\{ w_{k-1}(\delta_{m,n}) + \frac{\sqrt{\pi} \Gamma(k)}{4 \Gamma(k + 1/2)} \right\} & k \geq 3 \end{cases} \] (85d)

\[ v_{k}(\delta_{m,n}) = \begin{cases} \frac{1 - 2 \delta_{m,n}^{2}}{1 - \delta_{m,n}^{2}} & k = 0 \\ \frac{3 - 4 \delta_{m,n}^{2}}{\delta_{m,n}^{2}(1 - \delta_{m,n}^{2})} & k = 1 \\ \delta_{m,n}^{-2k} & k \geq 2 \end{cases} \] (85e)

with the approximation error smaller than \( \beta^{-3/2} \delta_{m,n}^{4} \).

4.2. Asymmetric modal self-impedance. Low frequencies \( \delta_{m,n} > 1 \)

Equations (51) and (52) have been used to rearrange Eqs. (84b) to the form valid within the low frequency band

\[ \theta_{m,n} \approx \frac{\pi v}{4 \delta_{m,n} \sqrt{\delta_{m,n}^{2} - 1}} + x_{m,n} \cos(2 \beta + \pi/4) + y_{m,n} \sin(2 \beta + \pi/4) \] (86a)

\[ x_{m,n} \approx \frac{1}{\sqrt{\delta_{m,n}^{2} - 1}} + u + \frac{v \arccosh \delta_{m,n}}{2 \delta_{m,n} \sqrt{\delta_{m,n}^{2} - 1}} \\
+ y_{m,n} \cos(2 \beta + \pi/4) - x_{m,n} \sin(2 \beta + \pi/4) \] (86b)

with the approximation error smaller than \( \beta^{-3/2} \delta_{m,n}^{4} \).
4.3. *Asymmetric modal mutual impedance. High frequencies*  
\[ \delta_{m,n} < \delta_{m,n'} < 1 \]

In the case of normalized acoustic impedance of the asymmetric membrane vibrations, a procedure similar to the one in the previous section has been conducted. The only difference is that the asymptotic expansions from Eqs. (74) have been used instead of those from Eqs. (48) and (46). Equations (54)–(56), (39) and (40) have also been used together with the methods of contour integral and stationary phase. Calculating the following integral

\[ \hat{I}_k = \delta_{m,n} \int_1^{\infty} dx \frac{dx}{x^{2k}(x^2 - \delta_{m,n}^2)\sqrt{x^2 - 1}} \]  

is necessary for this purpose for \( 0 < \delta_{m,n} < 1 \). Similarly as in the case of integral (81) it has been obtained

\[ \hat{I}_k = \frac{\arcsin \delta_{m,n}}{\delta_{m,n}^{2k}\sqrt{1 - \delta_{m,n}^2}} - \hat{u}_k(\delta_{m,n}) \]  

for \( k = 0, 1, 2, \ldots \) where

\[ \hat{u}_k(\delta_{m,n}) = \begin{cases} 
0; & k = 0 \\
\frac{1}{\delta_{m,n}}; & k = 1 \\
\frac{\hat{u}_{k-1}(\delta_{m,n})}{\delta_{m,n}^2} + \frac{\sqrt{\pi}\Gamma(k)}{2\delta_{m,n}\Gamma(k + 1/2)}; & k \geq 2
\end{cases} \]  

The normalized mutual radiation resistance and reactance assume the form of

\[ \theta_{m:n,n'} \equiv \theta_{m:n,m',n'} \approx X_{m:n,m'} \cos(2\beta + \pi/4) + Y_{m:n,m'} \sin(2\beta + \pi/4) \]  

\[ \chi_{m:n,n'} \equiv \chi_{m:n,m',n'} \approx -W_{m:n,n'} - W_{m:n',n'} + \frac{Z_{m:n,n'} \arcsin \delta_{m,n}}{\delta_{m,n} \sqrt{1 - \delta_{m,n}^2}} \]  

\[ + \frac{Z_{m:n',n} \arcsin \delta_{m,n'}}{\delta_{m,n'} \sqrt{1 - \delta_{m,n'}^2}} + Y_{m:n,m'} \cos(2\beta + \pi/4) - X_{m:n,n'} \sin(2\beta + \pi/4) \]  

within the high frequency band \( \delta_{m,n} < \delta_{m,n'} < 1 \) where it has been denoted

\[ W_{m:n,n'} \approx \frac{2\delta_{m,n'}}{\pi\beta} \sum_{k,l=0}^{N} (-1)^{k+l} \]  

\[ \times \left\{ \frac{\epsilon_{m,2k} \epsilon_{m,2l} u_{k+l+1}(\delta_{m,n})}{(8\beta)^{2k+2l+2}} + \frac{\epsilon_{m,2k-1} \epsilon_{m,2l-1} u_{k+l+1}(\delta_{m,n})}{(8\beta)^{2k+2l}} \right\} \]
within the low frequency band 1 < \delta_{m,n} < \delta_{m,n}'

The normalized mutual radiation resistance and reactance has been formulated as

\begin{align*}
Z_{m,n,n'} &\approx \frac{2\delta_{m,n}^2\delta_{m,n'}}{\pi \beta (\delta_{m,n}^2 - \delta_{m,n'}^2)} \sum_{k,l=0}^{N} (-1)^{k+l} \left\{ \frac{\epsilon_{m,2k}\epsilon_{m,2l}}{(8\beta\delta_{m,n})^{2k+2l+2}} + \frac{\epsilon_{m,2k-1}\epsilon_{m,2l-1}}{(8\beta\delta_{m,n})^{2k+2l+1}} \right\} \\
u_k(\delta_{m,n}) &\approx \begin{cases} 0; & k = 0 \\ \frac{u_{k-1}(\delta_{m,n})}{\delta_{m,n}^2} + \frac{\sqrt{\pi} \Gamma(k)}{2\delta_{m,n}\Gamma(k+1/2)}; & k \geq 1 \end{cases} \\
\chi_{m,n,n'} &\approx \frac{(-1)^m \delta_{m,n}\delta_{m,n'}\{Q_m^2(\beta) - P_m^2(\beta)\}}{\beta \sqrt{\pi} \beta (1 - \delta_{m,n}^2)(1 - \delta_{m,n'}^2)} \\
\gamma_{m,n,n'} &\approx \frac{(-1)^m \delta_{m,n}\delta_{m,n'}2P_m(\beta)Q_m(\beta)}{\beta \sqrt{\pi} \beta (1 - \delta_{m,n}^2)(1 - \delta_{m,n'}^2)}
\end{align*}

with the approximation error smaller than \beta^{-3/2}\delta_{m,n}\delta_{m,n'}^2.

4.4. Asymmetric modal mutual impedance. Low frequencies
1 < \delta_{m,n} < \delta_{m,n}'

The normalized mutual radiation resistance and reactance assume the form of

\begin{align*}
\theta_{m,n,n'} &\approx \frac{-\pi Z_{m,n,n'}}{2\delta_{m,n}\sqrt{\delta_{m,n}^2 - 1}} - \frac{\pi Z_{m,n,n'}}{2\delta_{m,n'}\sqrt{\delta_{m,n'}^2 - 1}} \\
&\quad + \chi_{m,n,n'} \cos(2\beta + \pi/4) + \gamma_{m,n,n'} \sin(2\beta + \pi/4) \\
\chi_{m,n,n'} &\approx -W_{m,n,n'} - W_{m,n,n'} - \frac{Z_{m,m,n'} \arccosh \delta_{m,n}}{\delta_{m,n}\sqrt{\delta_{m,n}^2 - 1}} \\
&\quad - \frac{Z_{m,m,n'} \arccosh \delta_{m,n'}}{\delta_{m,n'}\sqrt{\delta_{m,n'}^2 - 1}} + \gamma_{m,n,n'} \cos(2\beta + \pi/4) - \chi_{m,n,n'} \sin(2\beta + \pi/4)
\end{align*}

within the low frequency band 1 < \delta_{m,n} < \delta_{m,n}' with the approximation error smaller than \beta^{-3/2}\delta_{m,n}\delta_{m,n'}^2.

4.5. Asymmetric modal mutual impedance. Middle frequencies
\delta_{m,n} < 1 < \delta_{m,n'}

The normalized radiation resistance and reactance assume the form of

\begin{align*}
\theta_{m,n,n'} &\approx \frac{-\pi Z_{m,n,n'}}{2\delta_{m,n'}\sqrt{\delta_{m,n'}^2 - 1}} + \chi_{m,n,n'} \cos(2\beta + \pi/4) + \gamma_{m,n,n'} \sin(2\beta + \pi/4)
\end{align*}
Asymptotic Approximation of the Modal Acoustic Impedance of a Circular Membrane

\[
\chi_{m,n'} \approx -W_{m,n'} - W_{m',n} + \frac{Z_{m,n'} \arcsin \delta_{m,n}}{\delta_{m,n} \sqrt{1 - \delta_{m,n}^2}} \\
- \frac{Z_{m',n} \arccosh \delta_{m,n'}}{\delta_{m,n'} \sqrt{\delta_{m,n'}^2 - 1}} + Y_{m,n'} \cos(2\beta + \pi/4) - X_{m,n'} \sin(2\beta + \pi/4)
\]

within the middle frequency band with the approximation error smaller than \(\beta^{-3/2}\delta_{m,n'}^{-2}\).

5. Notes on the Numerical Analysis

The previous section presents a complete set of asymptotic formulas of the modal acoustic impedance. Now we will focus on the analysis of the approximation error and identify constraints of the results obtained.

In the numerical analysis of the approximation error in the asymptotic formulas it is convenient to rearrange the integrals from Eqs. (40) using the substitution \(x = \sin u\) for the radiation resistance and \(x = \csc u\) for the radiation reactance. This has given the following pair of integrals calculated within the finite limits

\[
\theta_{m,n'} = 2\delta_{m,n} \delta_{m,n'} \int_0^{\pi/2} \frac{J_m^2(\beta \sin u) \sin u \, du}{(\delta_{m,n}^2 - \sin^2 u)(\delta_{m,n'}^2 - \sin^2 u)}
\]

\[
\chi_{m,n'} = 2\delta_{m,n} \delta_{m,n'} \int_0^{\pi/2} \frac{J_m^2(\beta \csc u) \csc^2 u \, du}{(\delta_{m,n}^2 - \csc^2 u)(\delta_{m,n'}^2 - \csc^2 u)}
\]

A number of vibration modes are usually used in practical computational applications. In this paper, it will be a number of the modal acoustic impedance values. For practical reasons, the number of modes included must be limited, leading to the upper limitation in the range of the wave parameter \(\beta\) for the results obtained. Suppose we are interested in a number of the initial values of the modal impedance limited by the modal numbers in the range of \(m = 0, \ldots, 40\) and \(n = 1, \ldots, 30\). This choice is completely arbitrary. It results in the use of the matrix of \(40 \times 30 = 1200\) cosine modes and \(39 \times 30 = 1170\) sine modes and leads to the need to calculate \((40 \times 30)^3 = 1.728 \times 10^9\) modal impedance values. The latter number is very large and gives an idea of the computational complexity of the problem and the suitability of the use of asymptotic formulas presented. Obviously, it is unnecessary to give a graphic representation of all these values. However, it is important to determine the approximation error and the restrictions on the use of the asymptotic formulas. Presenting an actual measure of the approximation error will be particularly useful.

Since the number of modal impedance values within the specified range of modal numbers is very large only a few of them are shown in Figs. 2–4. The self impedance has been presented for \(m = 0, 20, 40\) and \(n = 1, 15, 30\) while the mutual impedance has been
Fig. 2. The acoustic impedance for $m = 0$: (a) self resistance (black) and self reactance (gray), (b) approximation error of self resistance — estimated (black) and theoretical (gray), key for lines: solid — $n = 1$, dashed — $n = 15$ and dotted — $n = 30$; (c) mutual resistance (black) and mutual reactance (gray), (d) approximation error of mutual resistance — estimated (black) and theoretical (gray), key for lines: solid — $n = 1$ and $n' = 2$, dashed $n = 15$ and $n' = 30$, and $n = 29$ and $n' = 30$.

illustrated for the same values of $m$ and for $n, n' = 1, 2, 15, 29, 30$ (the eigenvalues of the respective vibration modes are given in Table 1). The two most interesting cases have been selected from the multitude of the modal impedance values, namely those for which the two eigenvalues are as close as possible to one another — e.g. for the mode pair $(20, 29)$ and $(20, 30)$ and such, where the eigenvalues are as far as possible one another — e.g. for $(20, 15)$ and $(20, 30)$. All the other combinations of modes are something between the two cases mentioned and do not cause any further computational difficulties. It is necessary to remember that the number of nodal diameters of the two interacting modes must be the same — see Eq. (22). The self impedance has not been presented for $n = 2, 29$, in order to maintain the clarity of the graphs. The modal self resistance and self reactance curves have been presented together, because they can be easily distinguished. The self resistance assumes small values regardless the modal numbers, is characterized by some rapid low
amplitude oscillations for $\beta < \beta_{m,n}$, and tends to unity for $\beta \to \infty$ while the self reactance, generally, assumes significant values for $\beta < \beta_{m,n}$, and rapidly tends to zero and oscillates for $\beta \to \infty$. Both self resistance and self reactance assume positive values only whereas the mutual resistance and the mutual reactance assume the positive values as well as the negative values depending on the wave parameter $\beta$. These quantities assume significant values only for the values of $\beta$ close to the respective eigenvalues and tend rapidly to zero for the remaining values of $\beta$. The mutual resistance and the mutual reactance have also been presented together. Only the curves prepared using the asymptotic formulas have been presented. The respective integral curves have not been included for clarity’s sake. The numerical calculations have been performed using the general asymptotic formulas valid for any value of $m$. 

Fig. 3. The acoustic impedance for $m = 20$: (a) self resistance (black) and self reactance (gray), (b) approximation error of self resistance — estimated (black) and theoretical (gray), key for lines: solid — $n = 1$, dashed — $n = 15$ and dotted — $n = 30$; (c) mutual resistance (black) and mutual reactance (gray), (d) approximation error of mutual resistance — estimated (black) and theoretical (gray), key for lines: solid — $n = 1$ and $n' = 2$, dashed $n = 15$ and $n' = 30$, and $n = 29$ and $n' = 30$. 

Asymptotic Approximation of the Modal Acoustic Impedance of a Circular Membrane
The approximation error has been illustrated together with the modal impedance and estimated as

$$\Delta \theta \leq |\theta_I - \theta_A|, \quad \Delta \chi \leq |\chi_I - \chi_A|$$

where $\theta_I$ and $\theta_A$ are the resistance values calculated respectively from the integral and the asymptotic formula, and $\chi_I$ and $\chi_A$ are the respective values of reactance. The error value estimated this way for the reactance is very close to the respective error value estimated for the resistance for any value of the wave parameter $\beta$. Therefore, only the resistance error has been illustrated to maintain the graphs clarity. After completing a number of trial curves, it was found that the actual asymptotic computational complexity of the asymptotic formulas is $N^2 \geq (m/2)^2$. In practice, this means that the summation in the asymptotic
Table 1. Some sample eigenvalues of a vibrating circular membrane $\beta_{m,n}$.

<table>
<thead>
<tr>
<th>n \ m</th>
<th>0</th>
<th>20</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.40483</td>
<td>25.4171</td>
<td>46.6484</td>
</tr>
<tr>
<td>2</td>
<td>5.52008</td>
<td>29.9616</td>
<td>52.0161</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>46.3412</td>
<td>75.0763</td>
<td>101.155</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>90.3222</td>
<td>120.068</td>
<td>147.703</td>
</tr>
<tr>
<td>30</td>
<td>93.4637</td>
<td>123.253</td>
<td>150.964</td>
</tr>
</tbody>
</table>

Formulas can be limited according to the following scheme

$$\sum_{k,l=0}^{N} \rightarrow \text{Round}\{\frac{(m-1)}{2}\} \sum_{k,l=0}^{k}$$

(96)

where “Round” means rounding to the nearest integer. This indicates another limitation of the method, i.e. as the number of nodal diameter $m$ grows, the time necessary for numerical computations also grows as $m^2$ until it reaches the point for $m$ when the asymptotic formulas are providing the same time as the integral formulas. It has been tested for $m$ up to 40 that the asymptotic formulas are more efficient and the ratio of the time necessary for numerical integration and their asymptotic values are presented in Fig. 5. There are three curves that almost overlap each other for any $m$ as it is difficult to distinguish them. This indicates the fact that the computation complexity depends only on the number of nodal diameters $m$ and does not depend on the number of nodal circles $n$ and $n'$ of the vibration modes. For $m = 0$ the asymptotic formulas are approximately $10^4$ times faster than numerical.

![Fig. 5. The ratio of time necessary for numerical computations of the modal mutual impedance using the integral formulas $t_I$ and the asymptotic ones $t_A$ as a function of $m$. Key for lines: solid — $n = 1$ and $n' = 2$, dashed — $n = 15$ and $n' = 30$, and dotted — $n = 29$ and $n' = 30.$](image-url)
integration whereas for \( m = 40 \) they are only about 10 times faster. For higher values of \( m \) the asymptotic formulas become useless.

The rapid growth of approximation error can be observed below a value of \( \beta \) in Figs. 2–4. This value shows a strong correlation with the number of nodal diameters of the respective modes and this value is \( \beta \approx m \). This is due to the use of the asymptotic formulas from Eq. (76) and is related to the radius of convergence. Any further increase in the number of the asymptotic terms included cannot improve significantly the accuracy of the asymptotic results. Furthermore, the value of approximation error increases rapidly for values of \( \beta \) close to the respective eigenvalues. Since relatively many asymptotic terms have been used in Eqs. (76), so the approximation error is caused mainly by using the method of stationary phase from Eq. (80). This main error relates to the amplitudes of oscillating terms \( \mathcal{X} \) and \( \mathcal{Y} \) from Eqs. (85a). This leads to the theoretical value of approximation error

\[
\Delta \theta \approx \Delta \chi \leq \frac{1}{2\beta} \left( |\mathcal{X}| + |\mathcal{Y}| \right)
\]

This relation reflects well the error value in the range of \( \beta \), where the error reaches its small values. However, the error is underestimated for \( \beta \) close to the eigenvalues. Therefore the following improved relation has been introduced

\[
\Delta \theta \approx \Delta \chi \leq \frac{W(\beta)}{2\beta} \left( |\mathcal{X}| + |\mathcal{Y}| \right)
\]

where \( W(\beta) \) is the weight function. This function can assume some different forms. In this paper the following form has been selected

\[
W(\beta) = 1 + (h - 1) \left\{ 1 - \frac{1}{1 + \exp[-2(\beta - m)/\beta_w]} \right\} + 2(h - 1) \left\{ 1 - \frac{1}{1 + \exp[-2|\beta - \beta_{m,n}|/\beta_w]} \right\} + 2(h - 1) \left\{ 1 - \frac{1}{1 + \exp[-2|\beta - \beta_{m,n}'|/\beta_w]} \right\}
\]

where \( h \) and \( \beta_w \) are parameters to be determined empirically. This function is not free of faults and can be further improved. The following parameter values have been selected \( h = 10 \) and \( \beta_w = 7 \) causing that the value of approximation error is slightly underestimated or overestimated within the range of \( \beta \), where the error is less than \( 10^{-3} \). However, it well reflects the error value in other areas of \( \beta \) which was verified numerically for any modes described by the modal numbers \( m = 0, \ldots, 40 \) and \( n, n' = 1, \ldots, 30 \). This function can therefore be used to switch the computation algorithm using the integral formulas and the asymptotic ones within the range of approximation error values higher than \( 10^{-3} \).

6. Concluding Remarks

A set of asymptotic formulas of modal radiation resistance and reactance, self- and mutual, of a circular vibrating membrane has been obtained. These formulas are valid in almost the
entire band of wave factor $\beta$, except the lowest frequencies for $\beta < m$ and the frequencies close to coincidence for $\beta \approx \beta_{m,n}$. They may be useful to speed up the numerical analysis of vibroacoustic properties of flat vibrating elements, which can be modeled by means of a circular membrane reaching sufficient accuracy. Only the calculations for $\beta \approx \beta_{mn}$ and $\beta < m$ must be performed using integral formulas from Eqs. (94) or other approximations presented e.g. in Refs. 16, 17, 19–21, 31 and 41. Also it has been shown how to use the modal quantities presented to ensure some proper results for physical quantities such as acoustic power, acoustic pressure and vibration velocity including acoustic attenuation.

References

40. M. V. Fedoryuk, *Asymptotics: Integrals and Series* (Nauka, Moscow, 1987) [In Russian].